Angle Concept Formation in Elementary Age Children

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Abstract

This paper is a review of literature on the child's development of the angle concept. Children have difficulties understanding angles that stem from the lack of salience of the angle in general representations and particular angle situations, the wide variety of particular angle situations, and the fact that there is no single angle concept but rather multiple angle concepts. General frameworks for considering the development of the angle concept are discussed, as well as teaching experiments with concrete materials, Logo software and paper and pencil activities. It is found that development of a robust understanding of angles is a difficult task that requires extensive experience with a broad range of angle situations.

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Introduction

Length and angle are two of the most basic concepts in geometry. Children are able to recognize, reproduce, compare and measure lengths in a variety of situations from an early age, demonstrating a firm grasp on the concept of length. However, problems with the angle concept persist even into secondary school (Close 1982 and Mitchelmore 1983; as cited in Mitchelmore 1995).

Why should this be so? Consider the traditional pictorial representation of an angle in Figure 1.

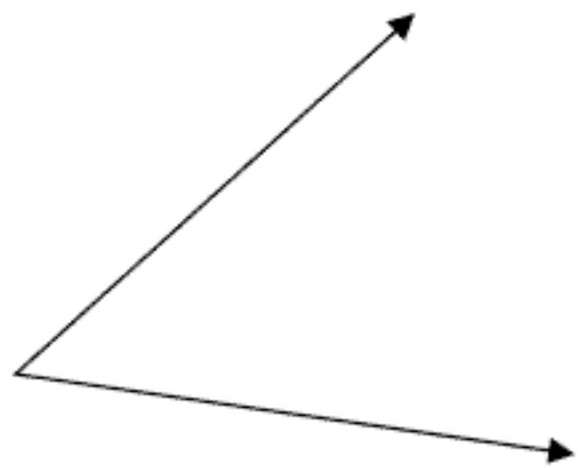

Figure 1. Traditional representation of an angle.

We can easily point out the lengths of the sides and the vertex, and we can indicate the space between the sides, but where is the angle? It's not something we can point to. In fact, mathematicians have struggled with different definitions of the term "angle" (Keiser, 2004) and a 1983 study found that German school textbooks provide definitions of "angle" that vary widely (Strehl 1983; as cited in Mitchelmore 1995).

Additionally, angles occur in a wide variety of physical situations that are not easily correlated (Freudenthal 1983; Mitchelmore 1995, 2000). For example, the same angle could be measured in the slope of a hill, the turning of a doorknob, the opening of a pair of scissors, or the height of the sun in the sky (angle over the horizon), but all of these situations are experienced as very different (compared, for example, to different situations under which one might measure length).

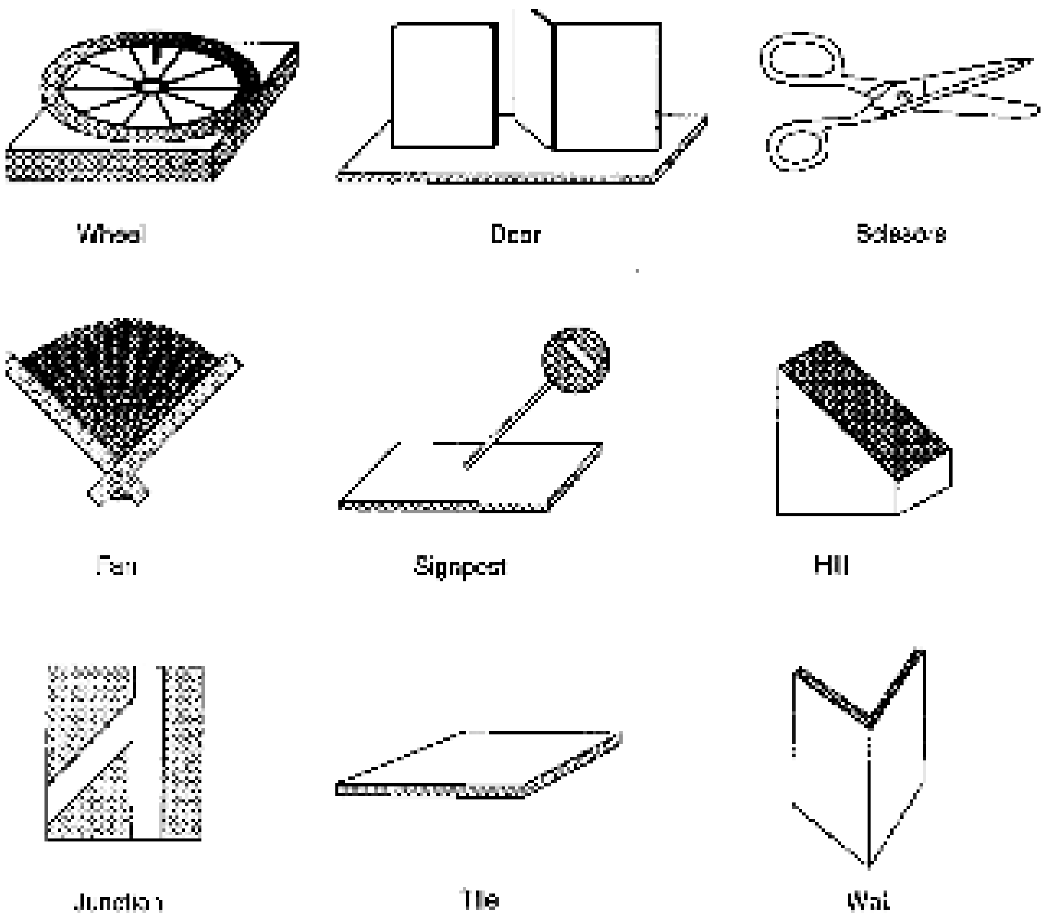

Figure 2. Various angle situations (from Mitchelmore 2000).

Montessori saw geometry as a way to help children make sense out of their visual experience. Helping them to develop an understanding and appreciation of form and dimension were as fundamental to Montessori as recognition and understanding of color (Montessori, 1917). Since her time, others have reported on the value of early experiences in geometry (Fuys,

1988) and in recent years the National Council of Teachers of Mathematics (NCTM) has put increasing emphasis on geometry in the elementary grades. Recent research on the angle concept can help support arguments for teaching geometry in the elementary grades and inform practice in the elementary classroom.

Background

As someone who is still in the process of developing a theoretical stance on this subject, I took a great deal of interest in reading about the theoretical frameworks of the authors in this review. Most researchers reviewed look to Piaget, van Hiele, or Freudenthal. It is interesting to note that Piaget was once the president of the Swiss Montessori Society, van Hiele was a former Montessori teacher, and Freudenthal (although he distinguished his approach from Montessori's) developed his theories in the Netherlands, where Montessori lived for many years (1936-1939 and 1947-1952) and had a tremendous impact on Dutch education. Papers that describe studies of angle concept development through the use of computer based (or technology rich) learning environments almost always contain references to Papert (a mathematician who studied with Piaget before he became one of the developers and main promoter of Logo).

My literature search revealed that the two most prolific (and most cited) researchers in the area of angle concept over the past twenty years or so have been Michael Mitchelmore (who also writes with Paul White and Anne Prescott) and Douglas Clements (who also writes with Michael Battista, Julie Sarama and others).

From Piaget, Mitchelmore (2000) takes the concept of *empirical abstraction*, the formation of concepts from everyday objects ("cup", "chair" and so on), though he sees this process as formation of a mental object that stands for a class of objects rather than the isolation of essential attributes of a set of objects as did Piaget (and many before him, back to Aristotle).

Mitchelmore also makes use of Piaget's distinction between formation of everyday concepts and logico-mathematical ones, which require *reflective abstraction* in which children construct not only new mental objects but also relations between them. This is the kind of abstraction required to recognize, for example, that slopes and turns are both angle contexts that can be compared through measurement of respective angles.

Mitchelmore also uses the idea of stages or levels of understanding used by Piaget and van Hiele. Mitchelmore (2000) identifies three stages of knowledge about angles:

- *Situated angle knowledge*. Knowledge of specific situations where angles are observed by the child, for example, a steep hill or the corner of a geometric inset.
- *Contextual angle knowledge*. More general knowledge that integrates related situations, for example, slope of hill with slope of roof, or turn of a doorknob with turn of a jar lid.
- *Abstract angle knowledge*. More general knowledge that integrates different angle contexts, for example slopes and turns.

Although Freudenthal is not specifically mentioned in Mitchelmore's work, it seems likely that Mitchelmore made use of Freudenthal's work that describes various angle situations (Freudenthal, 1983). In one of his papers, Mitchelmore (2000) lists Freudenthal as a reference although he does not mention him in the body of his paper.

Another important source of a theoretical framework for Mitchelmore is the work on situated cognition (notably Greeno, 1991, who in turn relies on seminal works by Brown, Collins & Duguid, 1989 and Lave & Wenger, 1991). From this work, Mitchelmore takes the idea of situated angle knowledge that is particular to a given situation.

Clements (2001) draws on Piaget and van Hiele for his theoretical perspective. He (Clements,1996) argues for geometry instruction with Logo by drawing on Piaget's idea that the child constructs understandings of space by moving through that space (in the case of Logo instruction, this also includes giving explicit instructions to a computer screen object to move forward or backward some amount or to turn left or right some amount).

Like Mitchelmore, Clements (1990) considers the development of the geometric understanding as something that occurs in stages. He modifies van Hiele's stages and proposes levels for study—*precognition* (Level 0), *visual* (Level 1; at this level children might say that they recognize a rectangle because it looks like a door), *descriptive/analytic* (Level 2; at this level, learners recognize a shape by its attributes, for example, a square has four equal sides and four right angles), *abstractional/relational* (Level 3; children can categorize objects hierarchically by attributes, for example they see that a square is a rhombus with the extra constraint of having four right angles), and Level 4 (this last level is the level at which learners can construct proofs). Clements also puts forward Papert's idea that the Logo environment enables children to make their intuitions into salient objects of reflection.

There were other interesting theoretical perspectives offered by some of the other authors including the angle concept at embedded in physical activity (Fyhn, 2006) and the development of the angle concept in light of the historical development of the concept (Keiser, 2004).

Disclosure

My own background includes experience as a Montessori teacher, a teacher of gifted classes in geometry (using both computer-based learning environments and manipulatives), and an educational software developer. I am a middle aged white male with a B.A. in Mathematics and an M.S. in Computer Science and have always enjoyed math. As a Montessorian, I am predisposed toward constructivist explanations of angle concept development and constructivist instructional approaches. As an educational software developer and Logo enthusiast, I am interested in finding ways to integrate computer-based geometry activities with other geometry activities in the Montessori classroom. However, through readings of relevant literature, classroom observation and experience with children in both Montessori and non-Montessori classrooms, I recognize that Logo or any other activity (on or off of a computer) needs appropriate teacher support (modeling of activity, setting expectations and so on) if it is to contribute to learning in the classroom. I also have found it important for children to have off-computer experience with angles as encountered in everyday life and in geometry textbooks.

Summary of Research Studies

In two of his studies, Mitchelmore (1995, 1998) Mitchelmore worked with small groups. In the 1995 study, he selected a group of 36 children with an average age of 9.6 years, from 2 schools in 2 grade 4 classrooms in western Sydney. Twelve of these students were interviewed (3 boys and 3 girls from each school). It's not clear how the original 36 were selected or how the 12 were selected for interviews.

In the study reported in 1998, Mitchelmore selected 36 children from grades 2, 4, and 6 from two Catholic schools in the north of Sydney, Australia and interviewed 6 students (3 boys and 3 girls) from each grade at each school. The average ages for children from grades 2, 4, and 6 were 7.1, 9.5 and 11.3 respectively.

Mitchelmore (2000) later reports that these earlier studies were inadequate to do all the analysis he needed due in part to the small sample sizes. In the study reported in 2000, he selected a sample of 192 children from grades 2 to 8. One hundred forty-four of the children were chosen from grades 2, 4 and 6 in each of six schools in Sydney, and 48 grade 8 students from two high schools. Other than age and gender, no demographic information was mentioned in any of his studies and I assume that this is because the available sample was pretty homogeneous. The 2000 paper is more cited by far than other Mitchelmore papers I reviewed.

All of the studies (including the huge 192 child study reported in 2000) were qualitative, using semi-structured interviews with a "fixed protocol" and neutral questions for clarification. Order of interview questions were "counterbalanced." The interviewer was a trained research assistant. A transcription of audiotape from interviews was coded by authors. Children were given various tasks to assess their level of angle knowledge (situated, contextual or abstract) by asking them to explain angle representations or relate different angle representations to one another. Task performance was scored and scores were analyzed for patterns in identifying similarities between situations or making global representations.

The main result of the Mitchelmore studies is that children do gradually subsume more and more different angle situations under the angle concept over time, which enriches their concept of angle. In one Mitchelmore (2000) study, children in grades 2, 4, 6 and 8 were given tasks in one-on-one interviews to show their ability to relate the objects in Figure 2 to a standard angle representation and to each other. A cluster analysis of coded results revealed that the clusters of correlations made by older children contained broader ranges of angle situations that the clusters of younger children. A pedagogical implication suggested by Mitchelmore is that children be given more opportunities to engage in tasks that help them make connections between different angle situations.

Mitchelmore later participated in studies that involved teaching experiments (Prescott & Mitchelmore, 2002; White & Mitchelmore 2003). In these studies, the authors break down angle situations into three clusters—2 line angles (corners of room or a geometric inset, intersecting roads, pairs of scissors), 1 line angles (doors, windshield wipers), and 0 line angles (the turning of a doorknob or a wheel). Treatment involved guiding students to abstraction from work with concrete materials (White & Mitchelmore, 2003). Students first developed *familiarity* with important features of particular angle situations. Next, different situations were examined for *similarity* through direct matching (physically superimposing the angle of one material on the corresponding angle of another) and indirect matching (using an intermediate angle, like a bent straw, to compare the angles of two other materials), and selective attention (pointing out the arms, vertex and opening of an angle in a particular material). Finally, children engaged in activities designed to help them abstract the angle concept from the concrete situations (children made abstract drawings of angles, were taught the terms "acute angle", "obtuse angle", and "right angle", then asked to describe the angle concept). The authors refer to this abstraction step as *reification.* Lessons were given by teachers after workshops given by the authors. Children received 8 lessons in the White study and 10 lessons in the Prescott study. In both studies, children made progress, but there were clearly difficulties in matching, measuring, drawing and describing angles, even after treatment.

The Clements studies of children's understanding of angles involved the use of the Logo programming environment, which is designed in part as a simulation of a "turtle" navigating a two dimensional plane. A study by Simmons & Cope (1990) makes it clear that children do not benefit from unstructured use of Logo. The authors interviewed 59 children between the ages of 9 and 12 who had had extensive experience with Logo for three month or more (some students had years of experience) but no explicit instruction. All of the children had trouble distinguishing internal and external angles, a distinction that is essential to understanding the creation of images with Logo (to create a particular internal angle, the turtle must move forward, rotate through the external angle, then move forward again). They followed up with a later study (Cope & Simmons, 1993) of 64 beginning Logo users aged 9 to 11 in which children were put into same sex pairs and given paper tasks and corresponding Logo based tasks. Both paper tasks and Logo tasks were given in random order, and the order of task sets (paper set first or Logo set first) was also random. Children were interviewed about the tasks by the authors who coded the children's responses. The authors found that children generally gave higher level responses when engaged in paper tasks.

Clements (2001) proposes that children can be helped to move from the visual level to the descriptive/analytic level of angle understanding as they make their intuitions explicit through structured interactions with Logo (e.g., they move from seeing a rectangle as something that "looks like a door" to something that is constructed with specific commands to move forward some amount, turn right ninety degrees, move forward another amount, and so on). Clements further proposes that children can be helped to transition from the descriptive/analytic level to the abstractional/relational by developing drawing programs that use variables.

In the Clements study reported in 1990, participants were twelve middle class children (four boys and eight girls) from the same fourth grade classroom. Children who returned a parental permission form were paired by gender and performance on the mathematics concepts and problem-solving subtests of the Iowa Tests of Basic Skills. Children from each pair were then assigned randomly to a treatment or a control group. The control group was given word processing lessons while the treatment group was given lessons in Logo by an experienced teacher. Both groups got two 40-minute sessions a week for 40 sessions. After about 25 sessions, both groups got regular geometry lessons with the Logo group getting extra lessons on

Logo turns and polygons. I guess the point of this was to show that the Logo treatment was better than no additional geometry work, but a better test would have been to have a comparison group that worked with concrete geometry materials or did paper and pencil geometry work while the treatment group was doing Logo.

The children in both groups were given structured interviews after preliminary lessons on either Logo or word processing, at about the 25 session mark, and after the treatment. Also after treatment the children were given a 10 problem posttest that was constructed by the researchers. The posttest "included tasks hypothesized to be dependent on either conceptual or procedural knowledge found in Logo programming or on the use of executive-level problem solving processes." (Clements & Battista, 1990, p. 361).

Not surprisingly, they found that children who did extra geometry work using Logo did better on geometry tasks and questions designed to test "knowledge found in Logo" than children who had no extra geometry work or Logo work. Clements and Battista (1990) report on ways that the Logo children demonstrate a higher order of understanding after treatment. For example, before treatment, all of the children reported that the size of an angle depends on the length of its sides. After treatment, all of the control group still thought this was true, while only two of the children in the treatment group thought so.

Another paper by Clements (1996) reported on a pilot study of 4 nine-year-old children (two boys and two girls) from a third grade classroom in a rural town and a field test conducted 2 third grade classrooms in an inner city school. The population in the field test was 85% African American (the rest were mostly white) with 80% of the students qualifying for Chapter 1 assistance in mathematics. Two of the students from the field test, a boy and a girl, were selected as case studies. Data was collected through interviews (pre and post), paper and pencil tests (pre and post), case studies and whole class observation.

This study (1996) was aimed at evaluating a curricular unit developed by the authors that includes tasks on and off of the computer. The lessons for both the pilot study and the field test were given by the authors. The focus of the curricular unit was development of the idea of angles as turns, in the context of constructing various paths. Some paths are constructed on the computer screen with Logo commands, and some are constructed as the children move physically through space. The goals are for children to recognize turns as changes in orientation (distinct from changes in position), to distinguish smaller turns from larger ones, to construct units of turn, to estimate turn sizes with units, to recognize different turns that yield the same change in orientation (e.g., turning right by 90 degrees yields the same result as turning left by 270 degrees). The field case studies show the development of the concept of turn (children learn to distinguish change in orientation from change in position and to use turns to construct paths), right and left directionality (learning to interpret "right turn" and "left turn" as "turn clockwise" and "turn counterclockwise" rather than "face east" and "face west", or generally figuring out which way to turn), and turn measure (learning to assign appropriate number of degrees to a desired turn). There was limited improvement in understanding for the field study. Clements (1996) concludes that changes in orientation are harder to understand than changes in position because they are less salient, that young children don't naturally connect static angles to turns, and that investigations of turns should be integrated into the K-6 curriculum.

Clements formed a hypothesis out of the 1996 study that students learn about turn measurement by integrating their understanding of "turn as body movement and turn as number" (Clements & Burns, 2000, p. 32). This hypothesis was tested when Clements (2000) was approached by two teachers who were looking for enrichment activities for 14 of their fourth grade students, who the teachers identified as high performers. Clements provided the lessons which teachers taught as in a pull-out program. Two boys, with no previous experience with Logo or angle measurement, were observed as case studies using field notes and videotape. As in the last study, a special version of Logo was used which was designed to makes turns more salient. This was done by having the turtle move more slowly, providing an option to show the initial ray of a turn along with a ray indicating current heading as the turn is made, a protractor tool to help learners to point the turtle to a given heading, and a dynamic geometry component to allow learners to directly manipulate drawn figures. It is not clear if the version of Logo in the 1996 study had all of the features of the version in the 2000 study. The work done by the children in the 2000 study (who had been identified as above average by their teachers) was compared with that done by the children in the 1996 study (who tested average or below average). The children from the 2000 study had no problems of omitting turns or amount of turn, distinguishing change in orientation from change in position, attending to left-right directionality and angle measure. Clements and Battista (2000) used a combination of qualitative analysis of children's talk and coding children's behaviors to put them into tables. They confirmed their original hypothesis and also found behavior patterns and narrative that indicated that students developed an internal image of dynamic movement as a result of experience they acquired in navigational activities that required moving their bodies.

While Clements demonstrated that children can make progress in their understanding of angles through activities with Logo, I didn't see any studies of his that directly answered the critique of Simmons & Cope (1993) that children can progress more quickly to a higher level of abstraction using pencil and paper rather than Logo, at least to the point where children recognize the relationship between turns and drawn angles.

An interesting study by Keiser (2004) compares the development of the angle concept in the child with its historical development. In this study, two classrooms were observed during a five week implementation of the first geometry work in the Connected Mathematics Project (CMP), *Shapes and Designs*. The teachers were early implementers of the program who were well acquainted with CMP and its methods, which include fostering an atmosphere in which children work together to construct meanings. The author used the historical development of the angle concept as a framework for analyzing the construction of the meaning of "angle" in the CMP classrooms. In particular, three topics were considered (Keiser, 2004, p. 288):

1. What Exactly Is Being Measured When Referring to the Size of Angles?
2. Can Angles Contain Curves?
3. Difficulties with Conceiving of 0°, 180°, and 360° Angles.

Keiser (2004) finds that her classroom observations and historical perspective agree with Freudenthal's (1973) assertion that there is more than one angle concept. Keiser (2004) also finds that students develop a more robust understanding of angles if they have the opportunity to explore multiple meanings of "angle" through a process of social construction.

Fyhn (2006) reports on a case study in which she explores the concept of angle with a twelve year old girl in the context of the girl's climb up a mountain peak in Norway that she did with family and friends. Rather than taping her interviews and interpreting transcripts on her own, Fyhn took field notes by hand and involved the girl in interpreting and correcting what she had written. Fyhn also drew on her own experiences climbing the same peak. Fyhn considers angles from three perspectives—(1) a geometric shape, (2) a dynamic motion, and (3) a measure.

In analyzing her interview of the girl, Fyhn found nine situations in which the girl referred to angles made by her body or by rock formations, and concluded that mountain climbing could serve as a context for developing the angle concept among learners who have an interest mountain climbing or find it exciting.

Conclusion

It is clear from the studies reviewed that the angle concept presents serious difficulty for elementary age children. We should allow a lot of time for explorations of various angle situations and not take for granted that children understand angles in general simply because they can work well with them in particular contexts. There are two important obstacles to children's understanding of angles: (1) angles are not salient in representations provided (Clements, 1996), so children confound the angle concept with salient aspects of representations (e.g., length of sides or space between sides), and (2) angles naturally occur in a wide variety of contexts that are not easily correlated (Mitchelmore, 1995).

In my experience working with gifted children from Montessori and non-Montessori schools, the Montessori kids in my classes were as likely as the rest of the kids to have problems with certain angle concepts, particularly those involving angles represented as turns. It would be useful to analyze activities using the theoretical framework provided by Mitchelmore (2000) to look for activity gaps in the Montessori curriculum and consider how gaps might best be filled.

Mitchelmore (2000) proposes that students be taught about different angle contexts by reference to everyday angle "situations," drawing on Greeno's (1991) idea of "situated knowing". This would involve children in tasks with "angle situations" such as those found in Figure 2.

The situated learning approach works well when learners need to understand a complex task in a particular context (Lave & Wenger, 1991; Brown, Collins, & Duguid), but a better way for presenting a basic concept such as angle may be found in the work of Maria Montessori. Montessori (1912) points out that before forming the concept "red," the young child is confused if the teacher points to a variety of objects (a truck, a shirt, a ball) and tells him that all of these objects are "red". Instead, Montessori provides the child with tablets that are identical except for color. In the first exercise with color tablets, the child is given a set of six tablets—two red, two blue, and two yellow. The child then has the opportunity to match the tablets. Through this activity, the child learns to see red, blue, and yellow and later spontaneously recognizes these colors in the environment. In practice, children sometimes need help recognizing objects in the environment as "red", but the preliminary exercises with the tablets prepare the child to do so.

Although it is not quite at the perceptual level of "red" or even "length," the angle concept is a basic concept that is likely taught best through stylized representations that highlight essential characteristics of angles in different contexts. For example, the standard Montessori elementary curriculum presents 2 sided angle contexts (1) corners (geometric insets, fraction circles, angles drawn as corners of polygons, intersections of lines, and two rays extending from one point) and (2) hinges (geometry sticks). Work with horizontal, vertical, and oblique lines fit into the category of the 1 sided angle context. There are currently no Montessori materials for 0 sided angles. Such material could be easily made, for example, a "dial" that fits into the Montessori material for measuring angles. A next step would be to help children relate these stylized representations through exercises in matching, measuring, drawing and describing the 2 sided, 1 sided, and 0 sided contexts. After this preparation, children could then be guided toward seeing angles in a wide variety of everyday situations. It would be interesting to compare this approach to teaching angles with Mitchelmore's situated learning approach.

I have also developed curriculum and accompanying software called Circular Reasoning (http://www.leonelearningsystems.com/circular_reasoning.htm) that takes a Montessori approach (using stylized representations of angles) to help kids relate different angle contexts and have done some preliminary testing with third and fourth graders. Circular Reasoning was also designed to discourage the use of lower-level strategies (often employed by beginners in Logo) that can slow development of higher level thinking about angles (Simmons & Cope, 1993). The use of these strategies in Logo may be due in part to the difficulty of backtracking in Logo (i.e., it is easier to iterate a little bit, check the orientation of the turtle, and then iterate some more than it is to try out a rotation, go back to the original heading, and then try out the rotation again with a different angle).

There are trade-offs in this approach. For example, the number and variety of constructions possible in Circular Reasoning is far smaller than the number possible in Logo. Also, Circular Reasoning does not lend itself to "body syntonic" reasoning as proposed by Papert (1980) and supported by Fyhn (2006). Still, it would be interesting to try to integrate Circular Reasoning software and curriculum with geometry activities that are already part of the Montessori classroom and with structured Logo activities that are provided after students have some amount of experience with disparate angle situations and can recognize turns as supplementary to drawn angles. More studies also need to be done comparing computer-based, manipulative and pencil-and-paper-based activities for learning about angles, evaluating the strengths and weaknesses of each to see how they can best complement one another.

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