Learning Math with Manipulatives

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Abstract

A standard set of variables is extracted from a set of studies with different perspectives and different findings involving learning aids in the classroom. The variables are then used to analyze the studies in order to draw conclusions about learning aids in general and manipulatives in particular.

Learning Math with Manipulatives

Manipulatives are a widely used part of the mathematics curriculum. The National Council for Teachers of Mathematics includes the use of manipulatives in its *Principles and Standards for School Mathematics* (NCTM, 2000), a document that has been used nationally for developing local mathematics standards. According to the National Center for Educational Statistics (NCES, 1993), during the 1992-1993 school year manipulatives were used every day for math or science in forty-nine percent of all public kindergarten classrooms. In the 1994-1995 school year, manipulatives were used in sixteen percent of all math lessons in US public eighth grade classrooms and in thirty-four percent of all math lessons in Japanese eighth grade classrooms (NCES, 1999). Research with manipulatives such as base ten blocks have been widely used, extensively researched, and yielded some remarkable results (Fuson, 1990).

However, research has not shown an unequivocal advantage to the use of manipulatives (Sowell, 1989). Manipulatives by themselves do not cause children to learn (Ball, 1992) or reform teacher practices (Moyer, 2001). Ball (1992) has argued that paper and pencil work as well as manipulatives for some purposes, and Fuson (1997) provides evidence for this in a follow-up study to her work with base-ten blocks.

This paper examines nine research studies related to the design of manipulatives or to teaching or learning with manipulatives. One of the studies (Kaminski, n.d.) involves images that are not manipulative, and another (Goldstone & Son, 2005) involves a computer simulation that is manipulated through a computer interface. These studies were included because they involve design elements relevant to the design of manipulatives.

This is the approach taken in this paper. Various studies are analyzed. Most of the studies were conducted over the past decade. All but one of the studies involved participants from preschool to middle school. A few studies were included that exclusively involve learning aids that are not manipulatives because they highlight important variables that are relevant to manipulatives or show how conflicting findings may result from studies that involve manipulatives. Most studies focus on features of manipulatives themselves or on interactions between learners and manipulatives, but studies are also included that give attention to the role of teachers.

Variables are drawn from the studies that relate to manipulatives themselves and to teaching and learning that make use of manipulatives to create a list of variables that is broader than the list considered by any individual researcher and which standardizes terms so that different studies can be compared more readily. An attempt is then made to explain the results of the studies in terms of this broader list. As a consequence, support for some findings are found in results that otherwise might seem unrelated or conflicting.

Basic Terms

Different studies examined in this paper use different terms. A few standard terms are established here and used in the text that follows so that studies can be more easily compared. A *learning aid* is defined as something that is present to at least one of the senses that a learner uses to develop understanding or proficiency (e.g., a textbook, a song to be learned, wine used in a wine-tasting class, a ruler, or a model of the solar system). A learning aid is *mutable* if it can be changed (as a piece of clay) or rearranged (as a set of Legos, or a collection of base ten blocks). A *manipulative* is a mutable learning aid. A *signifier* is something that stands for something else. This could be, for example, a word, a picture, or a model. A *referent* is something that a signifier stands for. For example, a referent for the numeral 2 is the quantity two. A referent for the word "Snoopy" could be the Peanuts character Snoopy or a Snoopy doll or a family pet named

Snoopy. A *representation* will be defined to be a mental construction such as a mental image that is used in thinking.

Summary of Research Studies

Fuson, Smith and LoCicero (1997) conducted a teaching experiment with first graders attending a K-8 school in a predominantly Latino neighborhood in which 87% of the students received free lunch. The school had two classes in each of grades 1-5. One of the classes in each grade was conducted in English and the other was conducted in Spanish. The Spanish speaking class ranged in size from 17 to 28 over the course of the year. The sample used for the study was the 17 students that were present at the beginning of the year. The English speaking class ranged in size from 24 to 28, and the sample used for that class was the 20 children present from December through June who were available for interviewing at the end of the year.

Teacher activities included leading whole-class discussions and problem-solving activities and individual and small group reflection, providing individual help, and organizing help by peers inside and outside of school. Links between number words, numerals and quantities were established and reinforced early and often. The teaching experiment was based on earlier work with base-ten blocks (Fuson, 1990). However, for this study, children used pencil and paper to draw dots and ten sticks to represent units and tens. This was done to provide a low cost alternative to base-ten blocks and provide a record of student work that teachers could later assess.

Data was collected through observation of class work and analysis of homework and the class work that was collected on some days. Teachers also provided input, and the English-speaking class was given one-item quizzes that were also analyzed. Children were interviewed at the end of the year. Interview questions were taken mainly from other studies so student performance could be compared with performance in those studies. The questions elicited children's understandings of (1) the relationship between quantities, spoken number names, and written number symbols for two-digit numbers, (2) place-value understanding and (3) two-digit addition and subtraction with exchanges. In the end of year interviews, children in both classes performed more like children in China, Japan and Korea than US children at their grade level or higher. Most of the first graders could perform two-digit addition and subtraction with exchanges and explain the exchanges, things that US first graders are generally not taught. Children in both classrooms outperformed Japanese and Taiwanese peers on some tasks.

Moyer (2001) conducted a year-long study with 10 public middle school teachers who elected to train at a 2-week summer math institute. The purpose of the study was to examine how and why teachers used manipulatives over the following school year. The teachers were selected from 18 of the teachers who took the summer training. Purposive sampling was used to obtain a range in teaching experience among participants (4-25 years with M=12.2 years). Forty classroom observations and 30 semi-structured interviews were used to collect data, as well as self-reporting by participants. Moyer found that the summer training led to an increased use of manipulatives, but no significant change in the way teachers taught. Manipulatives were used to reinforce ideas already learned or as a diversion rather than a way of helping children make sense of mathematics. Moyer attributes this to teacher beliefs, including the beliefs that "real math" is not fun and that work with manipulatives is not serious.

White and Mitchelmore (2003), conceptualizing manipulatives as affordances for abstraction, conducted a study to test the use of "familiarity, similarity recognition, and reification" in helping children abstract the angle concept. Their field study included 25 teachers of grades 3 and 4. White and Mitchelmore wrote 15 lessons that incorporated everyday objects (scissors, clock hands, rulers) and more stylized objects (pattern blocks, paper with folds radiating from center at 30 degree intervals). The different kinds of objects were not considered as conceptually distinct in the study, and no attempt was made to separate or sequence the use of them. White and Mitchelmore found that use of familiarity, similarity recognition and reification were helpful, but that learning of the angle concept was still problematic.

Goldstone and Son (2005) investigate different approaches to helping learners develop an abstract model of competitive specialization (the idea that parts of a system can self-organize without a central leader). Eighty-four undergraduates at Indiania University participated in the study in fulfillment of a course requirement. Students were divided into four groups working with computer simulations. Simulations could be represented as identifiable images (e.g., ants, apples and oranges) or as stylized images (dots instead of ants and blobs instead of apples and oranges). In the course of the simulation, ants are to self-organize spatially so that food is allocated evenly for the given territory and population. One group worked with the identifiable images throughout. A second group worked with stylized images throughout. A third group worked with stylized images and then identifiable images. The fourth group worked with identifiable images and then stylized images. The group that worked with identifiable and then stylized images performed best when tested on the concept of competitive specialization. The group that worked with stylized images and then identifiable ones performed second best. The group that worked only with stylized images did the worst.

Uttal et al. (1999) did a series of experiments with 2.5 and 3 year old children using two rooms. The first room was life sized, and the second room was a scale model of the first. Experimenters showed children where they hid a Snoopy doll in one of the rooms and children were given the task of finding the same dog (scaled to fit its room) in the other room. The purpose of the experiments were to see under what conditions the children could successfully use the scale model to “stand for” the larger room in order to locate the Snoopy doll. It was found that children had particular trouble when the relationship between the two rooms was not made explicit, or when the children had the opportunity to play with the miniature room before the experiment or if time went by between the time when the experimenter showed the relationship between the rooms and the time when the child searched. Further, talking about the relationship between the rooms was not enough, and children did better in the task if the miniature room was behind a window so they could not touch it, or if they used a photograph instead of a miniature room as the signifier for the larger room. From this, Uttal et al. (1999) concluded that manipulatives are useful if their link to referents is made early and often and if the manipulatives are not interesting objects in themselves.

Chao et al. (2000) distinguish two conceptualizations of manipulatives as (1) tools that help children form mental representations they can eventually use in place of manipulatives and (2) instantiations of an idea from which the child forms abstractions. They compare approaches to using manipulatives that correspond to these two conceptualizations: (1) structured activity with a single manipulative to make it easier for the child to internalize a particular representation of a number and (2) activities with varied manipulatives from which the child can abstract a general idea. Tiles shown in Figure 1 were used in a series of nine learning games with 157 kindergarten children participating over a five week period. The games were designed to teach recognition of number symbols, and addition and subtraction of single digit numbers. The children came from two classes in each of three schools in West Los Angeles, California. Each of the two classes were assigned randomly to either the structure or variety treatments. Chao, Stigler and Woodward found that children who received the structure treatment did better at addition tasks and somewhat better at subtraction tasks using nonfinger strategies than children with the variety treatment, but there was no significant difference in accessing quantitative meanings as measured by a Stroop-like test of numeric value recognition (see Figure 2).

Kaminski et al. report on an experiment to help children learn tasks analogous to modulo 3 addition. The purpose of the experiment is to compare two kinds of signifiers—(1) “relevant concrete” signifiers, i.e., that have features related to their referent (in this study, relevant concrete signifiers were iconic measuring cups filled to different levels to represent different numbers) and (2) “generic instantiations” that do not have such features (in this experiment, a stylized flag represented 0, a circle represented 1, and a diamond represented 2). The experiment is conducted with nineteen sixth-grade students from two middle schools in Columbus, Ohio. Children are randomly assigned to two treatments. They use statistical analysis of the results to find that sixth graders are better able to match corresponding elements in different situations and transfer knowledge about modulo 3 addition better when they use simple icons that are not related to the numbers they represent. They conclude that “children do not need a concrete instantiation to acquire an abstract concept.” (Kaminski, et al., n.d.).

Modulo 3 addition is a kind of addition that follows these rules: (1) add the two numbers together, (2) the result is the remainder of the sum and three. This yields the following modulo 3 addition table:

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

Modulo 3 addition is closed over the set {0, 1, 2}. In other words, the only numbers we can have in the modulo 3 addition table are 0, 1, and 2. The analogous signifiers used by Kaminski are shown in figures 3 and 4.

Meira (1998) investigates how middle school children interpret different kinds of instructional devices that embody the same idea. The participants are eighteen students from 3 eight-grade classrooms (8 boys and 10 girls aged 13-14) taught by 2 teachers in a public school in northern California. Before the treatment, regular classes were observed for three weeks so Meira could gather data on student background knowledge. Students then paired up according to their own preferences and then randomly assigned to one of three instructional devices. One device was a pair of winches in which all parts were clearly visible (see Figure 5). Each winch was attached to a block. The winches had different diameters, so they raised their blocks at different rates. Another device was a pair of springs that varied in elasticity (see Figure 6). The elasticity of the springs was not directly visible. Finally, there was a "number machine" (see Figure 7) which was a computer application that students used to input numbers and receive a pair of outputs. After school hours but in the school building, each pair took part in two 1.5 hour problem-solving sessions, which were videotaped. Questions about the devices were posed that required students to analyze data in terms of linear equations (equations of the form $y = mx+b$). Students used the devices to generate data which they analyzed to find patterns and underlying functions.

Zuckerman (2005) did a qualitative study in which 25 children aged 4 to 11 used two different manipulatives with embedded computation for a total of 40 hours, working both individually and in mixed age groups. The manipulatives were modeled after Montessori manipulatives (Zuckerman calls them "Digital Montessori-inspired Manipulatives" or Digital

MiMs, see appendix), in that they focus on presenting abstractions in concrete form and are intended for use by children at various levels of development. The purpose of the study was to see if embedded computation could be used to make a wider range of concepts accessible to children through manipulatives. Clinical interviews were used in which children were given various tasks to perform with the manipulatives. Children were questioned to ascertain levels of understanding. Zuckerman reported that children found Digital MiMs engaging, formed useful conceptions of rate, accumulation, feedback and probability, and were able to make meaningful analogies between simulations they built and real world events. Zuckerman also proposes specific guidelines for designing Digital MiMs (see appendix).

Discussion

The value of manipulatives, along with their proper design and use are all subjects of debate, and the articles summarized in this paper show some seemingly contradictory results. For example, Fuson (1990, 1997) showed that children made significant gains in math using signifiers of quantity such as base-ten blocks and dots and ten sticks, while Kaminski (n.d.) found that “relevant features” in a learning aid (likenesses between the appearance of the aid and the idea it conveys) interfere with learning. Mitchelmore (2003) found that children’s angle concept improved after hands-on experience with scissors, a fan, and other manipulatives, while Uttal (1999) demonstrated that toddlers do a better job at mapping tasks when they are prevented from touching a scale model of a room to be explored (or shown a photograph rather than getting their hands on the scale model). Many studies have reported that manipulatives are preferable to computer interfaces for helping children to develop math concepts, but Meira (1998) found the opposite to be true when helping children learn about linear equations.

An explanation of these various results requires a standard set of terms that accounts for the range of variables considered in the different studies. In the section that follows, some basic terms are defined. Next, some variables are drawn from discussions in the studies reviewed, and terms are standardized to facilitate analysis. Finally, the defined terms and variables are used in an analysis of the findings of studies in the preceding section.

Variables

In this section, an attempt is made to identify and standardize a set of variables relevant to the range of studies reviewed. Different researchers recognize and focus on different variables. There is also variety in the ways that manipulatives are used. In this section, an attempt is made to develop a list of variables involved in the studies reviewed. This list of variables will then be used to account for differences in results found in the different studies.

Among current researchers, the aspect of the learning aid that is generally given the most attention is the learning aid's level of abstraction. In the concrete-to-abstract dimension of learning aids, a distinction is generally made between (1) *symbols*, which have an arbitrary structure, and (2) *icons*, which do not (Ainsworth, 2006). For example, the Peanuts character Snoopy may represented by: (1) the written word "airplane", or (2) a picture of an airplane. The written word "airplane" bears no particular resemblance to an airplane, while the picture does. Symbols are considered more abstract than icons.

Many researchers feel that the level of abstraction is a critical variable for the study of learning aids, and seek finer and finer granularity of levels in order to clearly differentiate independent variables for study. For example, in considering icons of an airplane, we may consider a line drawing (abstract icon), a representational painting of an airplane (less abstract icon), or a photograph of an airplane (less abstract still).

In his work, Uttal (1999) uses the broader definition of symbol as “something that represents something else by association, resemblance, or convention, especially a material object used to represent something invisible.” (American Heritage Dictionary, n.d.). To avoid confusion between the two meanings of “symbol”, the term *signifier* is used in this paper to denote learning aids as “things that represent something else”. So, in this paper, a signifier can take the form of a symbol (the written word “Snoopy”) or an icon (a drawing of Snoopy or a Snoopy doll).

Uttal also distinguished between manipulatives as *thing-in-themselves* and manipulatives as signifiers. For example, when children aged 2.5 had the opportunity to play with the scale model of the life-sized room, they came to see it as a thing-in-itself and had trouble seeing it as a signifier to the life-sized room.

Manipulatives can be used to develop *explicit understanding* that can be transferred to a relatively broad range of situations. Manipulatives may also be used in an imitative way as tools to develop *proficiency* independent of explicit understanding. Either use may be appropriate depending on what is to be learned.

In the studies reviewed, there are two approaches to developing explicit understanding from experience. In one approach, manipulatives are used as *empirical instances* from which the child is intended to abstract a general idea.

Two distinct approaches are taken in two different studies to help children abstract from empirical instances. To help children develop the angle concept, White and Mitchelmore. (2003) guide learners through three stages of activities starting with everyday objects such as scissors and paper fans. The first stage develops *familiarity* with particular empirical instances by working with each one separately. For example, children might do work that involves opening scissors to various angles in developing the angle concept. In the second stage, children are helped to recognize *similarity* between different instances. For example, students may be asked to open a pair of scissors to match the angle made by a paper fan or the turning of a doorknob. In the final stage, *reification*, children are guided in activities that require abstraction (e.g., drawing abstract angles of a particular size and orientation, or describing the angle concept in their own words).

Goldstone and Son (2005) take another approach to abstraction from empirical instances. They sequence the different kinds of signifiers in an approach they call *concreteness fading*. In this approach, undergraduate learners begin their work with a life-like learning aid that gradually becomes more idealized. In one experiment, concreteness fading consisted of giving students a simulation with screen images ants, apples and oranges. After doing some work with the simulation, the screen images were changed to dots for ants and blobs for apples and oranges. Goldstone & Son also experimented with *concreteness introduction*, in which they started with dots and blobs which then changed to ants, apples, and oranges.

In contrast to empirical instances, manipulatives are also designed and used as *Platonic forms* (i.e., stylized or idealized forms of some idea). By manipulating this idealized form, children develop general ideas that can then be applied to specific instances. An example of Platonic forms is Montessori's pink tower, a set of ten cubes graded in size from 1 cubic centimeter (1x1x1) to 1000 cubic centimeters (10x10x10). The pink tower is used to help children develop an understanding of size.

Kaminski (n.d.) points out that learning aids can provide various amounts of information that are *relevant* or *irrelevant* to the learning task. Irrelevant information is most commonly found in empirical instances, but it is necessarily present in Platonic forms as well. For example, Montessori's pink tower has color even though it is intended to teach about dimensions. Montessori's color tablets have shape as well as color.

In summary, the following variables will be considered in the discussion of learning aids below: (1) level of abstraction (symbol or icon), (2) reference (signifier, referent, or thing-in-itself, (3) type of signifier (empirical instance or Platonic form), and (4) relevance or irrelevance of attributes.

It is also clear from multiple studies that guidance in the use of manipulatives is critical if learners are to benefit from their use. The following section describes the role of the teacher in using manipulatives.

Connecting experience and understanding: role of the teacher

The role of the teacher was critical in the teaching experiment conducted by Fuson and Briars (1990) using base ten blocks. Strategies employed included (1) helping the child maintain a tight link between blocks and written number symbols by having them immediately record results of operations after each step, (2) constant use of quantities (formed with blocks), spoken number names and written number symbols in order to strengthen associations between them, (3) allowing children to move away from blocks and work only with pencil and paper as soon as they feel ready, (4) monitor student performance carefully after they begin using paper and pencil exclusively to make sure they are not practicing errors, (5) begin immediately with four digit problems for addition and subtraction (or, if the child needs it, start with two digit problems and immediately follow two digit problems with four digit problems), (6) spend only 1-4 days exclusively working with place value and thereafter teach place value in the context of addition and subtraction problems and (7) for subtraction problems, do all exchanges needed before doing any subtracting.

Errors committed after treatment could frequently be corrected by simply reminding the children to “think about the blocks”. Uttal (1995) also found that toddlers have difficulty keeping signifier-referent relations in mind, but that reminding children of the signifier-referent relationship can help children overcome this difficulty. Even subtle reminders such as looking at the model were effective. Uttal cites the work of various authors that shows how subtle reminders of previous problems supports analogical reasoning from those problems. Chao (2000) also reports on the importance of teacher guidance in helping children establish and maintain links between signifiers and referents.

Fuson and Briars (1990) also report that the learning-teaching task is so complex that it is critical to enlist peers, parents and other adults to provide additional support. Fuson claims that child-invented strategies work better than problem-solving strategies imposed by teachers, citing Yackel (1995), and cites Cobb (1995) in stressing the need to teach peers to help by helping the learner solve problems using their own methods. Individual tutoring was a critical part of the treatment for some students. The classroom teacher in the Spanish-speaking classroom also helped children develop attentional capabilities with positive results.

As with Uttal (1995), Chao (2000) and Fuson (1990), White and Mitchelmore (2003) find that teachers can help children construct understanding by explicitly pointing out relations between manipulatives and analogous phenomenon. However, White and Mitchelmore are not focused on analogies between manipulatives and symbols or manipulatives and ideas. Rather, they focus on helping children see analogies between different manipulatives (e.g., a tilted sign, a paper fan, the corner of a pattern block, a turning doorknob) to help children abstract a general idea (i.e., the angle concept). While Fuson and Briars (1990) demonstrate that good teaching strategies can lead to good use of manipulatives, Moyer (2001) reports that ineffective use of manipulatives results from inadequate teaching strategies.

Connecting experience and understanding: role of the manipulatives

In general, empirical instances are more helpful than Platonic forms in developing proficiency independent of explicit understanding. For example, to learn how drive a car, it is best to learn with the car one will actually be driving in order to become accustomed to the specific controls changing gears, operating the steering wheel, brakes, and even windshield wipers and headlights.

Empirical instances can serve as aids to explicit understanding to the extent that analogous instances are visually identical, represent familiar objects, and have attributes that support the analogies to be made, and do not have attributes that interfere with analogies to be made. For example, in Montessori classrooms, a red ball and images of a red ball are used to represent verbs. The image of the red ball is familiar to children and evokes the idea of action. Every time a child sees a red ball in the context of grammar work, the red ball signifies a verb.

A problem with empirical instances is that learners tend to focus (at least initially) on surface features of an object or situation. For example, children may initially compare a car and a person in terms of surface features such as headlights and eyes rather than deep features such as engine and heart (Brown, 1992). Using Platonic forms, objects or situations can be created that reduce or background surface features that are not relevant to learning goals. For example, Montessori primary classrooms typically include tablets that differ only in color to help children form concepts of color.

This minimization of irrelevant surface features can make Platonic forms more useful than empirical instances when learners need to develop understanding they can use in new situations. Teachers help learners transfer knowledge from one learning situation to another by explicitly showing and talking about analogies between work with manipulatives and the new situations.

For example, Chao (2000)'s study suggests that Platonic forms are better than empirical instances for helping children develop nonfinger strategies for arithmetic operations. As with Fuson's work, the teaching experiments done by Chao included a tight coupling of numerals and quantities as a critical component. Children using empirical forms were more proficient using finger strategies. This proficient in finger strategies might be explained by the fact that the children using empirical forms failed to internalize operations as well as the Platonic forms group and so practiced finger strategies more.

Manipulatives may be more or less familiar to learners, and new manipulatives become familiar over time. Familiarity can support learning to the extent that it helps learners focus on features of a situation that are relevant to learning goals. Familiarity can also distract learners from learning goals to the extent that the manipulatives have surface features that are irrelevant to learning goals.

Goldstone and Son (2005) use familiarity to good effect in helping undergraduate students understand competitive specialization. They found that concreteness fading worked better than concreteness alone, abstraction alone, or concreteness introduction. As an explanation, Goldstone and Son propose that, compared to stylized images (dots and blobs), the recognizable images (e.g., ants, apples and oranges) in their experiment made a better "crutch" for grasping the meaning of the situation modeled and the relationships between its parts. This proposition could be tested by comparing concreteness fading and concreteness introduction (starting with stylized images and changing to recognizable ones) using both familiar and unfamiliar concrete situations. It could be that young children obtain a greater advantage from working first with Platonic forms if they do not have sufficient familiarity with relevant empirical instances.

In the study by White and Mitchelmore (2003), some of the manipulatives used by learners are previously known to the learners (e.g., a paper fan, a pair of scissors), some of the objects are scale models of known objects (e.g., a door on hinges, bends in a road), and some of the objects are unfamiliar (e.g., paper folded to make a 360 degree protractor). No attempt is made to treat learners separately with the different categories of manipulatives or to sequence the different kinds of manipulatives. Rather, children are given the opportunity to become familiar with the manipulatives through free play preceding treatment.

With their approach, White and Mitchelmore found that children made progress in constructing the angle concept, but progress was slow. Results are confounded by the fact that different kinds of manipulatives are not dealt with separately and there is no comparison of children who engaged in free play before treatment with a control group that did not play with the manipulatives before treatment.

For Goldstone and Son (2005), the appearance of the empirical instance served as an aid to understanding. Once a learner understood the behaviors of one ant and the fact that all ants acted the same way, it was easier to understand the behaviors of all ants. Further, because students could readily relate ants and food to particular behaviors, the students exhibited greater transfer to related tasks when concreteness fading was used rather than concreteness introduction. For White and Mitchelmore (2003) each empirical instance shared a particular attribute (angle) but had a different appearance. The appearance of the empirical instance presented a difficulty for students to overcome. Learners needed guidance to relate, for example, the angle made with scissors to the angle found in a corner of a pattern block, and it sometimes took years for learners to reliably relate different angle situations. Surface features of White and Mitchelmore's empirical instances might have obscured deep relationships between the instances, contributing to the slow progress that learners made in developing the angle concept.

Another way that manipulatives differ is the extent to which they lend themselves to off-task play. Uttal's (1999) work suggests that opportunities for free play with manipulatives makes it more difficult to see the manipulatives as signifiers for something else. In pilot testing, White and Mitchelmore (2003) found that children distracted from learning tasks with pattern blocks when they became engrossed in making different designs with the blocks. Interestingly, White and Mitchelmore's remediation for this problem was to give children the opportunity for free play with pattern blocks before treatment. So free play may hinder children's cognitive work (as in Uttal's study) or help that work (as with White and Mitchelmore), depending on context.

Differences between the two study contexts that might be important include (1) the treatment task in the Uttal study appears to be more structured, so the child has less opportunity for off-task activity during treatment, and (2) in the Uttal study, the move from free play to treatment required a shift in perspective on the manipulative from toy to signifier, while the move from free play to treatment in the White and Mitchelmore study required a move to greater focus on an aspect of the manipulative (the angles made by the corners of the pattern block) that children might have related to tasks they were trying to accomplish in free play (putting together different blocks to form patterns).

The tendency of learners to draw visual analogies can help or hinder learning as shown by Fuson (1990, 1997) and Kaminski (n.d.). Fuson finds visual analogies to be useful in the case of base ten blocks or dots and ten sticks. There are two important visual analogies between base ten blocks and the decimal number system. First, there is an analogous size relationship between the physical blocks and the numbers they represent (i.e., ten units are equivalent in size to a ten bar, ten of the ten bars are equivalent in size to a hundred square, and ten of the hundred squares are equivalent in size to the thousand cube). With dots and ten sticks, the size relationships are not strictly enforced because the child draws them by hand. However, there is at least a rough size correspondence. Second, the dimensions suggested by the bar, square and cube of the base-ten blocks (or the dots and ten sticks) correspond to the exponent of ten that is represented (the ten bar suggests a single dimension of size ten or 10^1, the ten square suggests two dimensions of size ten or 10^2, and the ten cube suggests three dimensions of size ten or 10^3.

On the other hand Kaminski et al. (n.d.) finds that visual analogies interfere with learning a task related to modulo 3 addition. The results of the experiments by Kaminski et al. (n.d.) might be explained by age of the learners involved. The children in Kaminski's study were middle school children, while Fuson (1990, 1997) worked with first graders. It might be the case that younger children are more likely to benefit from learning aids with visual analogies to the target domain. This idea could be tested by comparing transfer in younger children using materials with more or less relevant concreteness (for example, comparing learners using of Dienes blocks only with children using the stamp game).

It might also be that the measuring cup icons prompted learners to try to figure out a mathematical relationship between the numbers the icons represent, while the learners using "irrelevant concrete" icons (flag, diamond, circle) were focused on simply making the associations needed to obtain the right answer. If this were the case, the "relevant concrete" learners had a more difficult task. They were trying to draw analogies between the pictures they saw and arithmetical operations. The difficulty for them would be compounded by the fact that a full cup takes on the role of the additive identity rather than an empty one. This hypothesis could be tested by switching the full cup for an empty one and looking for transfer to math problems in addition modulo 3, and by testing both groups for transfer to problems using numerals instead of pictures (e.g., 2+2=1 in modulo 3).

Kaminski et al.'s (n.d.) study also fails to consider the fact that learners might benefit from guidance in drawing the appropriate analogies between learning aids and the target domain. For example, in Fuson's (1990, 1997) studies, visual characteristics of the ten blocks (or dots and sticks) were used in the course of activities that were analogous to the mathematical operations performed. For example, teachers spend time showing children the physical relationship between units and ten bars, helping children develop strategies for using the base ten blocks in addition and subtraction, showing them the links between quantities represented in base ten blocks and numerals, and reminding children of these links as necessary. Kaminski et al. (n.d.) provided "relevant concrete" icons with no suggestion that they could be used to figure out problems as done in modulo 3 arithmetic.

Transparency (i.e., the visibility of the inner workings of a device) can distract a learner when the learning goals involve the function of the device rather than its inner workings. This is shown by Meira (1998) who sees his winch device as most transparent and his number machine as least transparent and claims that his results show that opaque devices are better aids to learning about linear equations. Meira's results might be better understood if we look at his devices in terms of relevant and irrelevant attributes as described by Kaminski (n.d.). The winch and spring devices contain a larger number of irrelevant attributes (visible components) than the number machine. Also, as Uttal (1999) notes, learners have trouble seeing a learning aid as a signifier to the extent that they see the signifier as an interesting object in itself. In either case, the more transparent devices present the learner with something (irrelevant attributes or interestingness as a thing-in-itself) that make it harder for the learner to focus on the device as a signifier for linear equations.

None of the studies reviewed make direct comparisons between manipulatives and alternative learning aids. However, some comparisons are suggested. Ball (1992) claims there is no difference (at least in some cases) between doing arithmetic with beads or with circles one draws on a piece of paper. Fuson (1990, 1997) seems to support this idea. She did two separate teaching experiments, one with manipulatives and one without manipulatives. In the first teaching experiment (Fuson, 1990), children learned about place value and base ten arithmetic using base-ten blocks. In the second teaching experiment (Fuson, 1997), children learned the same subject matter but drew dots and ten sticks on paper instead of using manipulatives. Since the two studies were not compared directly, it is not clear if the results from either study were significantly different from the other. However, if there were no significant difference, it could be that advantages of using manipulatives were balanced out by the ability of teachers to assess student work when students used paper-and-pencil dots and sticks. Further research is needed to determine the relative advantages of these two learning aids.

Meira (1998) found that a computer application worked better than a physical device in helping students learn about linear equations. As stated earlier, the critical variable here could be the presence of irrelevant attributes in the physical devices. This could be tested by adding more devices to those in the original study, including computer interfaces with more irrelevant visual cues and physical objects with fewer irrelevant cues.

Uttal (1999) found that photographs or untouchable models made better learning aids than a touchable model for 2.5 year old children. This is an important finding. Just because a designer intends a manipulative to be a signifier (or a teacher presents it as a signifier), there is no guarantee that a child will recognize the manipulative as a signifier. The meaning of a manipulative (or any learning aid) must be constructed by the child. In Uttal's study, it may have been particularly difficult for the child to see the touchable learning aid as a signifier because the learning aid (a scale model of a room) had the appearance of a toy (a thing-in-itself). It could be that Platonic forms such as those used in Montessori classrooms are more readily seen as signifiers since they do not resemble everyday objects.

Conclusion

Research has not shown an unequivocal advantage to the use of manipulatives. Why not? There are three main reasons: (1) implementations vary, (2) the same results may be interpreted differently, and (3) there are not adequate studies comparing manipulatives with alternatives.

Implementations vary because of differences in manipulatives, teachers, and learners. Manipulatives may be more or less well designed. Different teachers may be better or worse at selecting and presenting manipulatives. Different learners may be better or worse at constructing the meaning of manipulatives and putting them to good use.

In the course of their work with manipulatives, students construct analogies between work with manipulatives and some concept or operation to be understood. For example, in work with base ten blocks, students must draw analogies between the operation of putting together arrangements of manipulatives and the arithmetic operation of addition. Manipulatives and teachers can *distract* students from making these analogies. For example, a student may be distracted if a hundred square is decorated in different colors or with a hundred different shapes rather than a hundred evenly spaced circles of the same color in ten rows and ten columns. A teacher may distract a student, for example, by inserting a time delay between a presentation of the base ten blocks to a student and the time that student can use the base ten blocks.

On the other hand, materials and teachers can *reinforce* analogies between materials and related concepts or operations. For example, many Montessori math materials use the same color code for elements of the material that represent the same category in the base ten system (i.e., elements representing units are green, elements representing tens are blue, and elements representing hundreds are red). Teachers can reinforce analogies by emphasizing the links between manipulatives, symbols and number words when presenting materials and later reminding students of these links.

Another reason for the differences in research results on studies of manipulatives is the different ways that authors conceptualize manipulatives and interpret results. White and Mitchelmore consider activity sequence (familiarity, recognition of similarity, reification) but not the added cognitive load irrelevant attributes (some manipulatives are empirical instances and some are Platonic forms, with no attempts to separate, order, or compare the two kinds of manipulatives). Kaminiski considers the relevance of attributes, but not the added cognitive load of treating images as signifiers for numbers (the measuring cup images with “relevant concreteness” use shading to imply number while the post-treatment test, like the “irrelevant concreteness” treatment, does not involve number, either implicitly or explicitly). Meira considers transparency but not the added cognitive load of considering the mechanics of a device rather than function only. Goldstone and Son find ways to capitalize on familiarity with everyday objects, but an important aspect of this work goes unmentioned—only a few kinds of everyday objects are displayed and objects that look the same have the same simple behaviors. Many of the studies take place over a short period of time and involve only a small set of learning aids.

Consideration must also be given to the way that work with a manipulative affects later learning experiences and how manipulatives are or can be designed to exploit earlier learning.

Finally, rigorous comparisons are needed between treatments using manipulatives and treatments that use alternatives to manipulatives. Fuson gives excellent accounts of two approaches to teaching mathematics, one using base ten blocks and one that uses less expensive manipulatives and makes more extensive use of pencil and paper. But the approaches are in two different studies. A rigorous comparison of approaches is needed.

Learning environments are highly complex, and any study of manipulatives in the classroom must account for wide range of variables. Four important variables were found in the studies analyzed: (1) level of abstraction (symbol or icon), (2) reference (signifier, referent, or thing-in-itself, (3) type of signifier (empirical instance or Platonic form), and (4) relevance or irrelevance of attributes.

A small sample of studies was considered in this paper. Consequently, there are many variables that have not been touched upon. A much broader meta-analysis needs to be done to uncover a range of variables wide enough to permit useful comparisons of findings and account for different results.

For example, no mention was made here about the role of the child in constructing understandings with manipulatives. Learning from experience requires intention, effort, and perseverance on the part of the learner, even with a competent guide. Manipulatives do not carry any meaning in themselves, and their meaning cannot be directly transmitted from teacher to learner. The meaning of manipulatives must be constructed by the learner. Regardless of the teacher's efforts, the learner must ultimately decide whether to use the manipulative as a toy, as a crutch, or as a tool for understanding. The relationship between the base ten blocks and our base ten number system may be immediately evident to an adult, but young children must construct this relationship., or ways that manipulatives can constrain or facilitate self-directed learning.

Another area that was not touched on in this paper is the way that manipulatives can facilitate or constrain the child's activity in ways that support self-directed learning.

In *Montessori: The Science Behind the Genius*, Angeline Lillard (2005) calls for further investigation into the efficacy of math manipulatives in general and Montessori's math manipulatives in particular. Analysis of Montessori's work could be improved with reference to a sufficiently broad set of standard variables. Further, Montessori's work could be used to uncover more variables. For example, Montessori is unique because of the breadth and depth of manipulatives she adopted, adapted, and developed, and how her manipulatives are visually related to one another and sequenced.

Different perspectives and different findings are useful in helping us develop better studies and clearer definitions of the variables at play in teaching experiments with manipulatives. In conduct such research and interpreting results, it is important to keep in mind that significant positive effects are not guaranteed for treatments involving manipulatives. Manipulatives can be helpful, harmful, or have a marginal effect, depending on how they are designed and used.

Manipulatives are not inherently good or bad. Neither are lectures, textbooks, chalkboards, pencil and paper, computers or any other learning technology. Investigations in the use of manipulatives must account for the complexity of learning situations, relationships between different learning activities, and learning over time when comparing learning with different manipulatives or learning with different manipulatives and alternative learning aids.

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Appendix

One of the Digital MiMs (Montessori-inspired Manipulatives) is called SystemBlocks. Each SystemBlock can represent a stock (a place where something can accumulate), an inflow, or an outflow. There are indicators on each of these blocks to indicate their state (how much stock has accumulated or the direction and speed of flow) and another kind of block to provide alternative signifiers system state (e.g., sound or graphs). These blocks could be used, for example, to simulate water in bathtub: water flows into a bathtub through a faucet (inflow), accumulates or drains from the tub (the stock), and flows out through the drain (outflow). When inflow is faster than outflow the tub fills, and when outflow is faster than inflow, the tub drains.

The blocks are generic signifiers. In other words, individual blocks do not look like faucets or tubs or drains or any everyday object. They are just blocks with inflow, outflow and accumulation indicated by arrays of LEDs. So the same blocks can be used to model "virus spread in a population, bank account savings growth from interest rate, CO_2 pollution growth from emissions and more" (Zuckerman, 2005, p. 860). Fifth graders used SystemBlocks to simulate dynamic accumulation.

FlowBlocks can be used give young children an impression of the relationship between probability and statistics and various distributions, and for tasks at increasing levels of complexity up to college level. As with SystemBlocks, FlowBlocks can be connected so that it appears that light is "flowing" from one block to the next. FlowBlocks also include blocks that are "probability forks"—blocks from which light can "flow" to either the left or the right. Probability forks have a slider mounted on the surface. When a child moves the slider on the probability fork from left to right, the probability that light will "flow" through the right fork increases, and vice versa. The child can also modify rules that apply to a particular block. For example, the child can add a rule to make light go faster when it passes through a particular block.

The following guidelines were used in the design of the MiMs: (1) the MiMs should be used to model generic structures rather than specific ones, (2) the MiM-created models should not look like real-world objects, (3) tight coupling between MiM components and their meaning should be maintained, and (4) MiM models should relate to a variety of real life situations to encourage analogies.

Table 1

Summary of Studies

First Author	Year	Aspects Studied (author language)	Sample Size	Age	Results
Uttal	1999	Use of scale model and other aids as maps	multiple samples	age 2.5 & 3 years	Links between signifiers and referents must be constructed and maintained. Manipulatives can be helpful or harmful.
White	2003	Sequence by familiarity, similarity recognition, reification	25 teachers (P)	3rd & 4th grade	Guidance helps, but angle concept is still problematic
Goldstone	2005	Sequence by level of abstraction	84 (P)	undergraduate	Concreteness fading most effective
Chao	2000	Single v. varied manipulatives	157 (P)	kindergarten	Platonic forms better for abstraction
Kaminski	n.d.	Relevant/irrelevant attributes in learning aid	19 (P)	6th grade	Relevant attributes hinder learning
Meira	1998	Transparency	18 (P)	Middle school	Transparency hinders learning
Zuckerman	2005	Montessori design approach	25 (P)	age 4-11	Montessori-inspired manipulatives promote motivation, concept formation, and useful analogies
Fuson	1997	Guidance in linking verbal, written and manipulative signifiers for number	37 (P)	1st grade	Guidance can help children construct and maintain links (appropriate analogies), move away from reliance on manipulatives.
Moyer	2001	Teacher beliefs	18 (P)	Middle school	Training increases use of manipulatives but does not reform practice.

Figure Captions

Figure 1. Tiles used in Chao et al. (2001) study of structure and variety conditions for use of manipulatives in conceptualizing the numbers from zero to ten. Section (a) has structured tiles to afford development of images for mental manipulation. Section (b) shows the range of patterns used for variety conditions to enable abstraction from a range of experiences. Section (c) shows further variety was introduced by using different shapes for the patterns in (b) that are enclosed by squares. Reprinted from Chao et al., 2001.

Figure 2. Stroop-like test of numerical interference used by Chao et al. (2001). Starting around third grade, children respond more slowly when the physically larger numeral has a lower value.

Figure 3. Signifiers used by Kaminski (n.d.) in treatments.

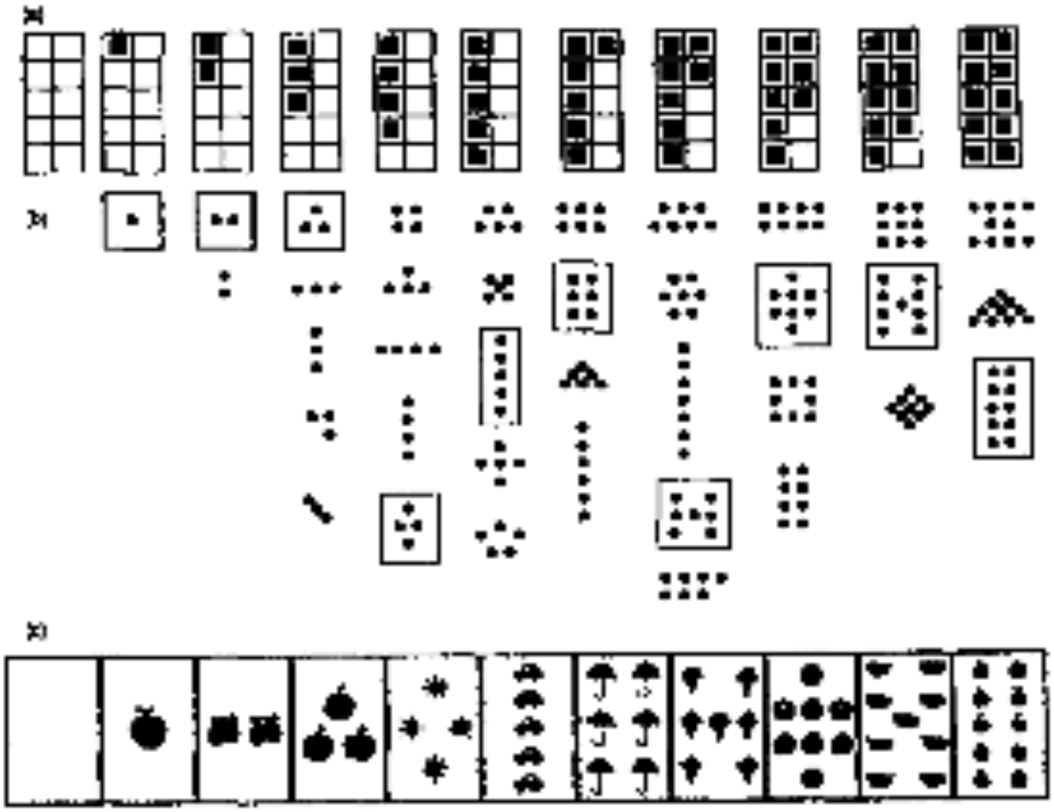
Figure 4. Signifiers used by Kaminski (n.d.) in assessment.

Figure 5. The winch device from Meira (1998) can be represented by the linear equation $P_f=CN+P_0$*, where* P_f *is the final position of the block (the blocks are shown as a black circle and a white circle on the figure), C is the circumference of the spool, N is the number of times the spool is turned, and* P_0 *is the position of the block before the spool has been turned (N=0).*

Figure 6. The springs device from Meira (1998) has two springs of different elasticity. This device can be represented with the linear equation $L_f=KW+L_0$*, where* L_f *is the final length of the spring, K is the coefficient of elasticity, W is the weight on the spring, and* L_0 *is the length before any weight has been added (W=0).*

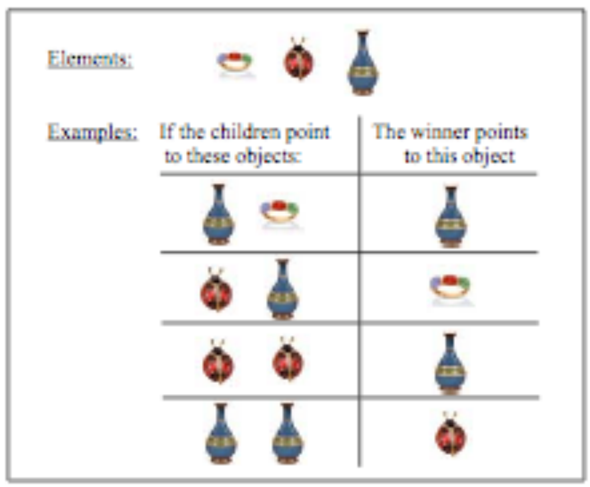
Figure 7.The number machine from Meira (1998) can be represented by the linear equation O=mI+b, where O is the output, m is the slope of the equation used by the number machine, b is the y-intercept used by the number machine, and I is the input number between 0 and 5 that is

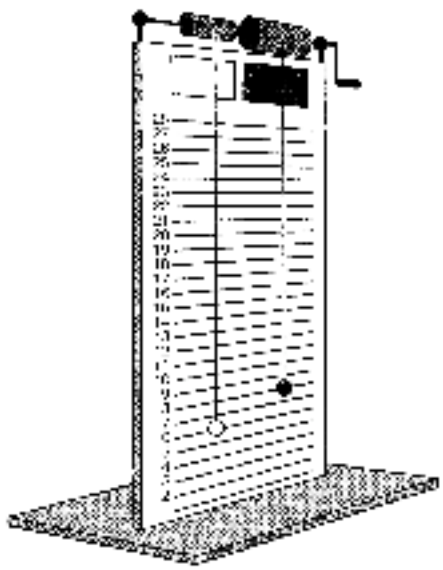
typed in by the student. In the figure, the student has just input 5. The output for 5 for the black side is 28. The output for 5 for the white side is presumably being calculated.

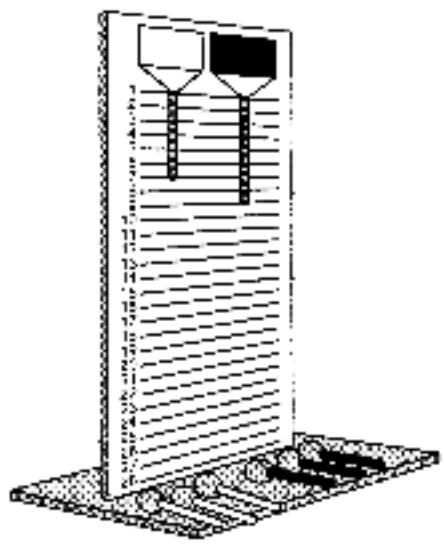


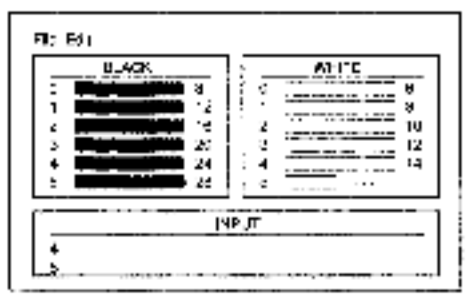
5	8
4	7
3	8
6	1
7	9
6	5
1	3
9	8
4	3
3	5
4	2
6	7
1	2
3	2
5	4

	Relevant Concreteness		No Relevant Concreteness	
Elements				
Rules of Commutative Group:				
Associative	For any elements x, y, z: $((x + y) + z) = (x + (y + z))$			
Commutative	For any elements x, y: $x + y = y + x$			
Identity	There is an element, **I**, such that for any element, x: $x + \mathbf{I} = x$			
Inverses	For any element, x, there exists another element, y, such that $x + y = \mathbf{I}$			
Specific Rules:	 is the identity		 is the identity	
	These combine	Remainder	Operands	Result
				
				
				


Elements:
Examples:
If the children point to these objects:
The winner points to this object






BLACK
INPUT